PEL Pensamiento Educativo. Revista de Investigación Educacional Latinoamericana
2015, 52(2), 172-188

Pontificia Universidad Católica de Chile
Facultad de Educación

Cognitive Analysis of Numerical Sequences: Information Processing and Genetic Epistemology

Análisis cognitivo de la secuencia numérica: procesamiento de la información y epistemología genética

Catalina María Fernández Escalona

Facultad de Ciencias de la Educación, Universidad de Málaga, Spain

Abstract

This research discusses the ways in which children interpret and build knowledge of ordinal numerical sequences. The logical-ordinal relationships that exist between terms of numerical sequences constitute the theoretical reference for this study. The analysis of how that knowledge-building occurs is carried out through different cognitive interpretations based on two very distinct models: the Piagetian model and information processing. The first model focuses on the logical structure of the seriation underlying numerical sequences, while the second looks at conceptualization and functionality as part of counting. The main result is a new way to study the development of numerical sequences in children by means of logical-ordinal relationships. The two models considered address the development of natural numbers. This study analyzes the development of numerical sequences by means of logical-ordinal relationships through the lens of these two models. Neither the Piagetian model nor the skill-integration model has specifically focused on these relationships, which means that the research conducted here is unprecedented and original.

Keywords: ordinal, logical ordinal relationships, number, number sequence

Post to:
Catalina María Fernández Escalona
Departamento de Didáctica de la Matemática, de las Ciencias Sociales y de las Ciencias Experimentales
Facultad de Ciencias de la Educación.
Campus de Teatinos s/n, Código Postal 29071, Málaga, España.
Email: cfernandez@uma.es
Thanks to the members of the Research Group HUM 205: Child Education and Training of Educators, Universidades de Andalucía.

ISSN: 0719-0409 DDI: 203.262, Santiago, Chile
doi: 10.7764/PEL.52.2.2015.22

Resumen

Esta es una investigación que trata sobre la interpretación y construcción del conocimiento ordinal de la secuencia numérica en el niño. El referente teórico elegido es el de las relaciones lógicas ordinales que se dan entre los términos de la secuencia numérica. El análisis de cómo se da esa construcción del conocimiento se realiza a través de las distintas interpretaciones cognitivas desde dos modelos bien distintos: modelo piagetiano y procesamiento de la información. En el primero se trata la estructura lógica de seriación subyacente a la secuencia numérica y en el segundo se analiza la conceptualización y funcionalidad como componente del conteo. El resultado principal es que hay una nueva forma de estudiar el desarrollo de la secuencia numérica en el niño mediante las relaciones lógicas ordinales. Los dos modelos considerados tratan sobre el desarrollo del número natural, y en este trabajo se analiza bajo el prisma de esos dos modelos el desarrollo de la secuencia numérica mediante las relaciones lógicas ordinales. Dichas relaciones no han sido objeto específico de estudio ni en el modelo piagetiano ni en el modelo de integración de habilidades, lo cual significa que la investigación aquí realizada es totalmente inédita y original.

Palabras clave: ordinal, relaciones lógicas ordinales, número, secuencia numérica

This paper discusses the analysis of two major lines of research regarding the development of the number in children from the perspective of logical-ordinal relationships existing between the terms of the numerical sequence. Taking into account studies on this development (Brannon, 2002; Fernández & Ortiz, 2008; Fuson & Hall 1983; Mix, Sandhofer, Moorey, & Russell, 2012; Slusser & Sarnecka, 2011), two major lines of research have emerged, which have been shown in studies on the teaching and learning of this concept: on the one hand, the *Piagetian logical model* (Piaget & Szeminska, 1964) and, on the other, the *skills integration model (information processing)* (Gelman & Gallistel, 1978).

Comparing the two models in term of the common object of study, the development of the number in children, from the perspective of knowledge we have the evolutionary psychology of Piaget; in this model, the evolution of child development is often more demanding, being concerned with cognitive maturity (Piaget, 1981; Serrano, 2008); while the other approach favors precocity and quantification of the given (Gelman & Gallistel, 1978; Sarnecka & Gelman, 2004).

While certain relationships exist between the two models, it should be underlined that both theoretical frameworks are not directly comparable given that the former refers to the conceptual construction and operation of the number, while the second allows the creation of a counting model, as shown in Figure 1. In spite of this, they do have a common denominator, namely the study of development of the number in children, but it must be restated that they are not directly comparable because the logical-Piagetian model addresses the conceptual construction and operation of the number and rejects practical or empirical counting, while in the skill-integration model, counting models are prioritized and the number is treated as a quantifying operator through the action of counting.

If the theory of information processing is taken as a reference framework, analysis of the numerical sequence appears to be a component of counting; while if logical-Piagetian theories are used as reference, the numerical sequence is studied under the operational structure of seriation in relation to inductive reasoning (Castro, Cañadas, & Molina, 2010).

It has to be said that in this study, the analysis is carried out addressing the two models in the development of the number, as shown in the diagram in Figure 1. This analysis was performed using the author's reflections for each of the models based on empirical research (Fernández & Ortiz, 2008) on the logical-ordinal relationships between the terms of the numerical sequence, with the common thread of these reflections being comparison between the two models in terms of the development of the numerical sequence in children and treating the number as an element in a sequence or series, ignoring its cardinal aspect, while the ordinal aspect becomes relevant.

Fernández and Ortiz (2008) conducted an empirical research with 27 children aged between three and six to study the ordinal use of counting. The aim was to prove that the origin of counting is subject to logical-ordinal relationships that develop in the process of mental construction of natural numbers. Many of these relationships have been considered in research on child psychology (the Piagetian model and information processing) without the proper importance because, the cardinal aspect of natural numbers has been considered as support for the ordinal aspect.

In the aforementioned investigation, semi-structured interviews were used to gather information in an organized manner. These interviews include certain key questions that all respondents have to answer. All of the interviews were recorded on audio and video. The aim is to observe how children express themselves regarding the logical-ordinal relationship of «immediately next» between consecutive terms of the numerical sequence by the comparison seen between them through the relationship established by a given serial correspondence, The child must establish this correspondence between one series with a criterion of simple alternation and the series of the numerical sequence. In accordance with the responses obtained, the support tasks in the interview allow a scalogram to be established to organize individuals into different levels from an evolutionary point of view.

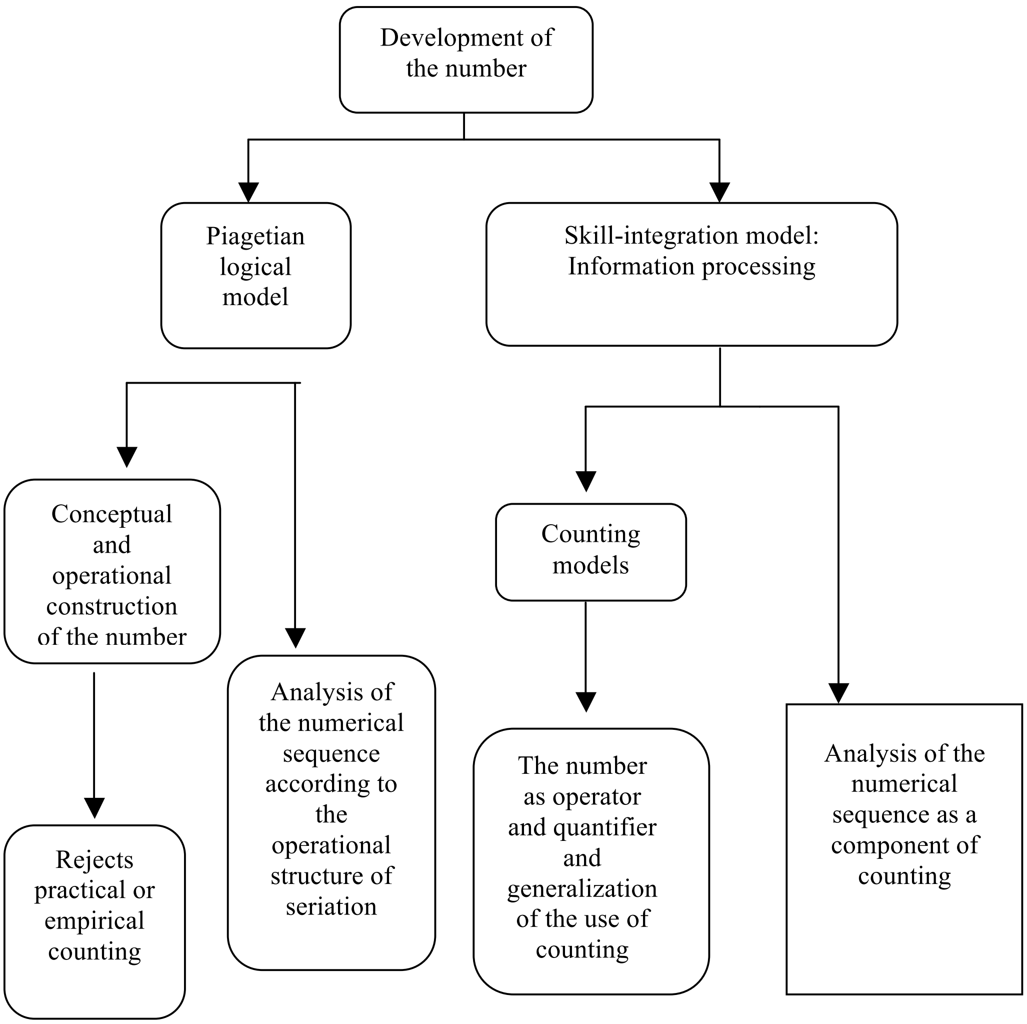

Figure 1. Development of the number according to the two theories: Logical Model and Information Processing.

We place ourselves in information processing and, from that perspective, we will study the numerical sequence as a component of counting.

Having located the numerical sequence in relation to counting as a more overall procedure, we turn to analyze this within procedural theories, first studying their conceptualization and then the ordinal functional characteristic. In the aforementioned analysis, we will insert certain specific qualifications that are not included in this theoretical framework, regarding the logical components that interrelate each and every one of the terms of the sequence, with the logical-ordinal relationships between the terms of the numerical sequence being the main point of all the reflections and analysis in this article.

Finally, we will study the logical structure of the seriation underlying the numerical sequence, analyzing the specific elements of this structure that occur in the numerical sequence.

Numerical sequence and the act of counting

It has been discovered that children can handle the numerical sequence from a very early age (Fuson & Hall, 1983), but it is possible that they only know that the counting sequence is composed of numbers and these always have to be repeated in the same sequence (Brainerd & Gordon, 1994; Sánchez, 2013), without this inferring a certain conceptual comprehension in accordance with which the order of the terms remains constant or each number is unique.

Knowledge by rote in recitation must exist, and this refers to conceptual comprehension. This comprehension implies two basic aspects: on the one hand, is the order in which the terms appear in the recitation, which is an unvarying property, meaning that the numerals are linked by a «next» relationship. On the other hand, there is the anti-symmetric property; the relationship of immediate next in mathematics fulfills this property, because if a term *a* is immediately followed by the term *b* and *b* is immediately followed by *a*, then *a* and *b* are equal, and this property ensures that the elements are not repeated.

Fuson and Hall (1983) carried out a longitudinal transversal study of children from two to eight years of age to study the *acquisition and production* of the sequence of numerals. These two phases overlap each other at a certain time, since a long time is required to acquire and consolidate the sequence. For example, the process of establishing relationships between the first terms can begin while the sequence lengthens; in other words, the first fragment of the sequence can be in the phase of preparation, while the other end is in the full acquisition phase.

During the acquisition phase, learning of the conventional sequence is done and the child begins to apply it in counting situations. The sequence works as a unidirectional overall structure consisting of the following parts: a stable and conventional initial part; then a stable unconventional fragment; and the final part consists of fragments that are neither conventional nor stable.

In the production phase, the links between the elements become stronger and adjacent terms can be issued at the margin of the overall sequence. Each term can be used as a support element to remember that which is immediately before or after. This phase, according to Fuson and Hall (1983), is subdivided into five levels: string, unbreakable chain, breakable chain, numerable chain, and bidirectional chain.

Fuson and Hall's levels can be reinterpreted based on the logical structure of seriation and logical-ordinal relationships as follows:

1. The relationship between the terms of the numerical sequence is anti-symmetrical. That is, each term of the sequence occupies a unique place and is uttered only once. In the performances of children in a counting situation or merely in a situation where the sequence is recited, this scheme is illustrated if the children utter the sequence without repeating any term in it (in terms of recitation) and do not count one place twice (in terms of counting situations).

2. The numerical sequence is a succession of nexts that start at one. That is, the numerical sequence is not uttered as a whole, but rather there is differentiation between the terms because each one of them, except the first, is uttered after another. It is determined that each term has a unique next, but until that time, for children, these nexts always appear when the sequence begins at one. As for the performances of the children, this scheme can be seen if they are able to establish a one-to-one correspondence between the objects in the countable set and the number sequence as opposed to the «superficial gesture» of children who utter the sequence as a whole.
3. The succession of nexts is a characteristic that remains despite any division of the numerical sequence. The fact that one term follows another is independent of the first term chosen for the start of the count. Therefore, it is a property that is conserved regardless of the initial reference. In children's performances, and following a logical order of evolution in accordance with Fuson and Hall's levels, this aspect is seen when children are capable of counting from any term without having to start at one.
4. Finite section in the succession of nexts. The first element is considered that which is before all others and the last is the one that comes after all others. In children's performances that take into account this logical-mathematical scheme there is the power to count or utter the sequence from any term *a* up to any other term *b*, considering *a* and *b*, respectively, as first and last.
5. Different senses: ascending and descending in the succession of nexts. In the utterance of the sequence, both in an ascending and descending sense, various logical schemes can be seen:

- ✓ Both the next and previous of any element can be determined.
- ✓ Analogously to the succession of nexts starting at any term *a*, there would be a succession of previouses.
- ✓ Just as knowledge is acquired of «all the previouses», the kind of «all the previouses» is also obtained.
- ✓ In the same way that the position of a term can be determined taking as reference a previous position through progressive counting, the position of a term can be determined taking as reference a later position through regressive counting.

To conclude, we have included a summary table in which notions are indicated that teachers should take into account for performance in the classroom, addressing the levels of Fuson and Hall (1983).

Table 1
Levels of proficiency of the numerical sequence and consequences for performance in the classroom

Proficiency in the numerical sequence in the act of counting	
Levels	For performance in the classroom
1. String. The succession of terms is produced beginning at one and the terms are not well differentiated.	✓ Recitation of the numerical sequence by the value of the recitation itself. ✓ Songs that stimulate rhythmic and linguistic learning of the sequence.
2. Unbreakable chain. The succession of terms is produced beginning at one and the terms are well differentiated.	Typical activities: ✓ Build a set with a given number of elements. ✓ Find the nth element in a series.
3. Breakable chain. The succession can begin at any term *a*.	Activities that involve these situations: ✓ That following a number lower than 10. ✓ Continue a 10. ✓ Following a number with change in the 10.
4. Numerable chain. To count n terms beginning with *a*, another term *b* must be given as a response.	Activities with these situations: ✓ Count from *a* to *b*, considering all possible cases (same ten, change in the 10, etc.). ✓ Count from *a* to n in terms considering all possible cases (a and n lower than 10, between 10 and 20, between 20 and 30, etc. Count from 10 by 10s, 11 by 11s, etc.).
5. Bidirectional chain. The succession can go up or down from any term. The direction can be changed easily.	✓ Study all possible cases that can occur between *a* and *b* and between *a* and *n.* ✓ Previous and next of a number *a.* ✓ Compare *a* and *b* in these cases: - *a* and *b* in the same 10. - *a* and *b* in different 10s.

In a different order of things and following with the numerical sequence as a component of counting, we will focus on an aspect that we have not mentioned so far, which is the conventional or social character of the terms.

The question addressed at this point is to see if any «list» can be counted or whether, on the contrary, the «number sequence» is irreplaceable.

Regarding this question, there are various positions: to Gelman and Gallistel (1978), with the principle of stable order, or to Wagner and Walters (1982), who distinguish a «strong» and a «weak» form of the same principle, any list works, while authors such as Fuson and Hall (1983) argue that the numerical sequence is irreplaceable. Given this discussion, one must focus on the use of the numerical sequence compared with any other list, for several reasons: (a) it is learned early at school for cultural reasons and (b) the numerical series has its own structural characteristics that no other series has, unless it is subject to structural isomorphism (i.e. a bijective application between two systems that maintains the structure) with a sequence of 10 digits.

Fuson and Hall argue that the sequence of numbers is irreplaceable on four points: (1) the information provided by some studies (Bermejo & Bermejo, 2004; Fuson, 2000, which addressed children's proficiency in counting) which show that the students conceive the conventional list of numbers as an irreplaceable tool; (2) the fact that they judge the counts done by a puppet, where the puppet does not properly apply the counting sequence, as erroneous (this puppet is a doll used by Fuson and Hall, 1983 in one of their tests, where if a child makes an erroneous count, the puppet makes the same erroneous count and the child then detects that the puppet has made a mistake); (3) the conventional stable segment that heads all sequences uttered by students (even from two and a half years old), as this reflects the attempts made by those students to learn «the special list» for counting; and (4) the precedence of stable sequences in understanding cardinality.

There are other positions in which the numbers are linked by a relationship of next and not by solid structures (Fernández, 2010; Fernández & Ortiz, 2008; Muldoon, Lewis, & Towse, 2005). We must not fail to consider the contributions of Song and Ginsburg (1988) with their studies on the nature of the elements in the counting sequence. In these cultural studies it is observed that, in almost all languages, numerals up to 100 are produced through a system based on a rule combining units and tens. The numerical sequence has a generation system that replaces rote learning starting at 10.

In light of this, positions will be adopted in this regard when the numerical sequence is analyzed from the perspective of the logical structure of seriation, according to which the terms will be linked by the relationship of next, which in turn will lead to the operational construction of the underlying structure in the systematization of the sequence. The numerical sequence is analyzed as a component of counting; later, it will be analyzed with the Piagetian paradigm, but it must be clarified that, in both models, the research context followed is that of logical-ordinal relationships existing between the terms of the numerical sequence.

Authors such as Song and Ginsburg (1988) defend rote learning of the sequence, at least regarding the section from 1-10, as they understand that early numerical ability of students is due to the creation of habits and, therefore, they propose that the mechanical application of counting should be gradually modified by the comprehension of counting.

Figure 2 shows a schematic summary of this study.

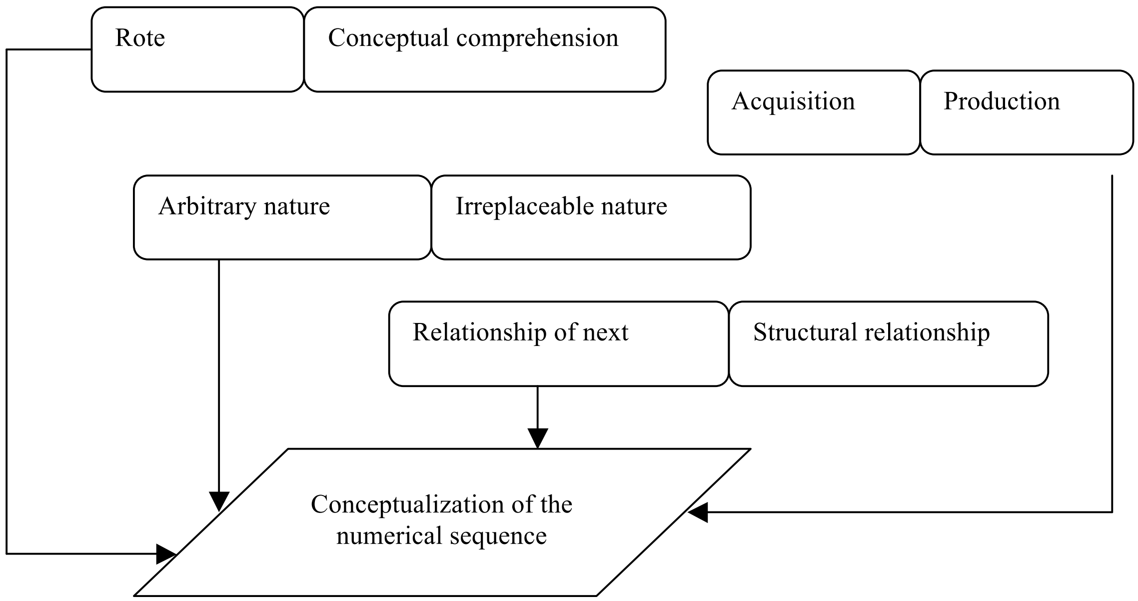

Figure 2. Conceptualization of the numerical sequence contextualized in Information Processing theories.

Functional character of the numerical sequence in an ordinal context

Once the numerical sequence has been conceptualized as a component of counting in information processing theories, we complete the study by analyzing the functional nature.

The ability to count does not have a goal in itself; it is a very powerful strategy in the mathematical development of the student. With regard to counting, there are two lines of research: on the one hand is conceptualization and, on the other, its functional nature. The first focuses, above all, on how children understand and coordinate each of the components and their evolutionary course, while the second seeks to determine the ability of children to solve problems.

Parallel to research focused exclusively on the study of counting, there are other studies in which the aim is to determine the ability of students to solve problems where counting is used as a procedure (DeSoto, 2004).

The numerical sequence has been analyzed in the previous section as a component of counting and, in turn, this has been treated as a process in itself in the field of conceptualization mentioned above. In this section, we make a shift in this treatment and look at the functional value; thus, it is a reflection on how logical connections are manifested between numerical terms through their use.

The intention is to use the functional value of counting to establish ordinal relationships between numbers. Therefore, whenever we speak of the logical components underlying the numerical sequence, this refers to the ordinal relations between their terms and the logic underlying the cardinal aspect of the natural number will not be studied in this paper. The object of study of this paper is the comparison of approaches between the Piagetian model and information processing, but this comparison is produced with the reasoning of the logical-ordinal relationships existing between the terms of the numerical sequence, which leads us to an ordinal study of the sequence without working on the cardinal aspect.

The cardinal aspect of the natural number has been discussed in depth in procedural theories about the functionality of counting, but similar treatment has not been seen regarding the ordinal aspect. Most papers in the literature (Gelman & Gallistel, 1978; Klahr & Wallace, 1973; Schaeffer, Eggleston, & Scott, 1974) related to this, take cardinality as mental support. It is a quantitative ordinal comparison: each number in the sequence represents the cardinality of a set and then comparison is made between the terms. This view is part of the logical construction of Bertrand Russell (1982): the natural number is defined through finite cardinals and then the relationships of order are defined (for a formal definition of the natural number and numerical sequence see Fernández, 2010).

The studies of Bermejo and Lago (1991) are along these lines, analyzing the functional nature of counting in ordering tasks. It seems that this is a useful way to avoid purely rote knowledge; the authors support the idea that if the quantity does not affect a task, children are not capable of establishing ordinal comparisons between the numerals because they take the form of «more/later» and «less/before». Consequently, tasks in which the way that two numbers are compared that represent two cardinal numbers obtained prior to the count seem to be common: these are the usual tasks of comparison of magnitudes, such as tasks in which the child has to decide whether they want to eat five or seven cakes in which they have previously been presented with one set of five cakes and another of seven cakes.

In contrast to these studies, the author of this article defends the hypothesis of optimal tasks in which exclusively ordinal relationships become apparent, which consists of the resolution of specific problems regarding the ordinal number, that is, to determine the position of a term in a series that has previously been considered a countable set to continue being an orderly set (Fernández & Ortiz, 2008).

The tasks in which the ordinal position of an element must be determined in a countable set through the numerical sequence, solely and exclusively evaluate the ordinal powers of the system through its use. These tasks are relevant to this study in comparison with others in which recitation of the sequence can be rote, and if the student is merely asked to count objects, it would be difficult to assess whether or not logical relationships between the terms have been established; or rather, if habitual tasks of comparison of magnitudes are proposed, the «isomorphism» between the cardinal and ordinal is being assessed (i.e.

«*a* is greater than *b* if, and only if, *a* is after *b*») and this moves away from the objective, which is none other than the comparison of any two terms of the sequence through the ordinal position that they have within it (Sánchez & Fernández, 1999).

The numerical sequence as a series in the Piagetian sense

Until now the numerical sequence has been studied in the context of the action of counting, that is, the model of skill-integration will now be analyzed taking into account the other model in question: the logical-Piagetian model.

Study of the numerical sequence in the Piagetian framework means working on the logical structure of seriation underlying the numerical sequence. In the diagram below (Figure 3), the passage from serialization to systematization of the sequence is shown schematically, understanding the boxes in the intermediate areas as serial skills that the child must apply to reach that point. In Piagetian terminology, the term «systematization of the sequence» means 'reaching the operative success of the series', and the operative success, in this study, is the establishment of ordinal relationships between the terms of the numerical sequence.

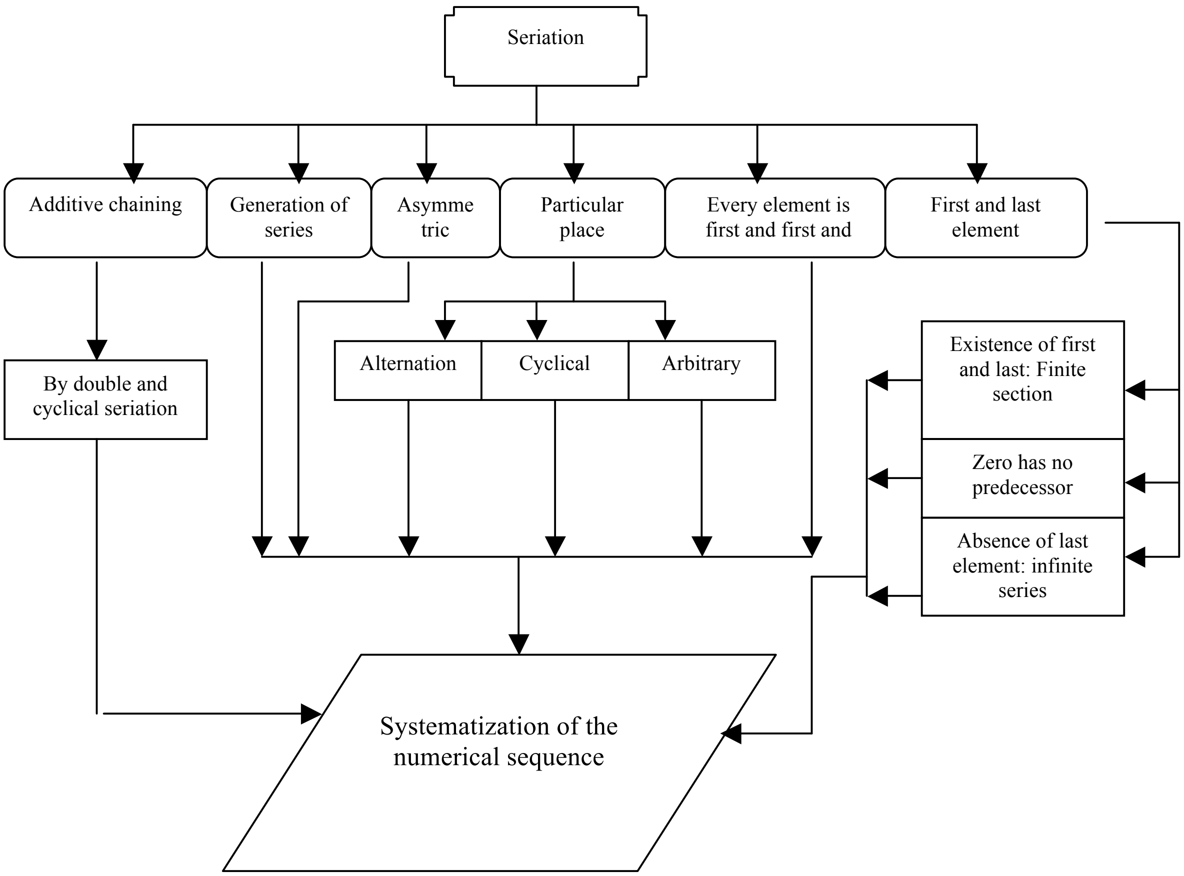

Figure 3. Systematization of the numerical sequence in the context of operational seriation.

We will address the serial skills presented in the following points. The transformation of the psychogenesis of seriation to the development of the numerical sequence are reflections of the author on the basis of qualitative empirical research on logical-ordinal relationships between the terms of the numerical sequence (Fernández & Ortiz 2008).

Asymmetric relationship

This refers to the comparison using ordinal terminology: previous, next, etc., of any two terms. This refers to noting the differences between two elements of the series relative to their ordinal position.

Additive chaining

This skill refers to the process of construction of a *succession of nexts*. One element is followed by another element and this element by another, and so on successively until the whole series is completed. «Additive chaining» is a recursive procedure from which the «succession of nexts» is obtained (Piaget & Inhelder, 1976).

The application of these schemes (additive chaining and succession of nexts) relies on the understanding that the first part of the sequence (0 to 9) is a *cycle* based on which, and with a rule of combination, the whole series of natural numbers is generated. In turn, this rule leads to the application of *double seriation*, which consists of seriation using two criteria, for example, width and height, which leads to a dual input provision. In the case of a numerical sequence with double seriation, one reaches the table of 100, in which each row represents 10 and each column represents the units (Fernández, 2010). Looking at the numerical sequence in this way facilitates learning of mental arithmetic just as suggested by Galvez et al. (2011), for example, adding 10 to a number is to move to the next row and the same column in the 100 table.

Additive chaining activities raise questions such as: continuing a given series, chaining elements, figuring out the next number, etc. To plan these activities one must take into account the genesis of knowledge and study the evolution experienced by children when given seriation tasks. Considering the psychogenesis of seriation, there are three stages of maturation until operative success: (a) *absence of seriation*: children are incapable of maintaining the criterion of the series and faced with tasks such as stringing alternate red-blue beads, they change the criteria, focusing on figural aspects; (b) *seriation by «trial and error»*: is the ability to seriate successfully through empirical trial and error; this action leads to successful completion of the series, but it is done by trial and error, and students are unable to anticipate an outcome; it is intuitive seriation, the additive chaining is only understood in function of the whole perceived series; (c) *operative seriation*: is where the operative success appears, the student is able to anticipate the series and does so using a systematic method, successfully expressing the generalization in problems of successions (Cañadas, Castro, & Castro, 2012).

The evolution that follows additive chaining goes through a first stage of arbitrary seriation in which there is solely a juxtaposition of terms and an absence of a law of succession; this is followed by intuitive seriation carried out by empirical trial and error which does not entail ability to anticipate, a systematic method, etc. and is characterized because while they are perceived, relationships between the terms occur, but they cease to exist in the mind of the student when they are destroyed; ending the operative success of the third stage in which the inverse relationships «greater than» and «less than» occur, which imply the possibility of developing the series in both directions.

Taking these actions to the numerical series and starting out with the child being proficient in the 1 to 10 section, the author concludes that the student: (a) fails to repeat the sequence from 1 to 100, for example, but is capable of reproducing small sections of it, (b) is able to count from 1 to 100 but with help in changing tens, and (c) achieves operative success, knowing a systematic approach to repeat the numerical series, knowing that when the numbers that begin with 1 are «exhausted», that is, when they get to 19, the following number starts with 2 and links this to all the numbers in the cycle (and follows with 20, 21 ... 29), and when this ends they should continue with 3, producing 30, 31 ... 39, etc.

Therefore, just as we have indicated, the psychogenesis of seriation can apply to the development of a sequence.

First and last element

This skill warns that in certain finite series there is a first and last element. The «first is before all others» and the «last is after all others». In order for a finite series to have a first and last element, it has to be «well ordered», that is, there has to be «good ordering» and «total order» (Russell, 1982).

The activities associated with these schemes are of the following type: construction of a series giving the first and last element; starting the series from a term *a* and ending it at *b*; that is, saying n terms based on *a* (Fernández & Ortiz, 2008).

Assimilation of these two characteristic elements of any finite series with a linear diagram shows the start of operative success, since identifying the elements *a* and *b* as first and last involves: (a) noting the differences between each of these elements and all others; (b) using terms that describe a series in a comparative sense compared with the use of the same terms in a purely labeling sense, and thus indicating that *a* is the smallest of all and *b* is the biggest, or in ordinal language to say that *a* is before all others and *b* is the last; (c) making use of the comparative series in both directions, because if a child has to stop at the last item *b*, they must recognize the term *k* as prior to this to know that what comes after *k* is the last, which means making simultaneous use of the concepts «previous» and «subsequent», developing the series in both directions.

Every element is first and last

A term in a linear series is the last element of all those that precede it and the first of all those that succeed it. This skill is inferred from the *ordinal series*[1], in which a total order relationship is involved; from all of them we can say that any element is greater than all those before it and lower than all those subsequent to it.

Identification of any of these terms presupposes the operative success of the performance of series, since this determines a systematic method for the construction of such series; this being consistent with placing the first element in first place, and then placing the first of the remainder, etc. Then, in each step, the element that is placed is simultaneously treated as first and last (Piaget & Inhelder, 1976).

Particular place

Every element occupies a particular place in the series. This refers to the ability to discover the position occupied by a given element applying different serial schemes: (a) alternation: a particular element is between two elements of the opposing type; this reveals certain important properties, such as an even number lies between two odd numbers, just as every odd is between two even numbers; (b) cyclical: by knowing the position of each of the elements that make up the cycle, the previous and subsequent elements of all others can be determined; (c) arbitrary: this means figuring out the position occupied by any term and observing how the description of that position is carried out (Piaget & Inhelder, 1976).

Generation of series

This is the process of generation of series referring to ordinal criteria. It describes a process of generation of the additive numerical series based on the sequence of natural numbers in the following way:

1. Construction of the series S_1. We make a serial correspondence between the numerical sequence[2] (which we will call S) and the alternation: yes-no-yes-no-yes-no-yes-no… We now consider the series of the sequence corresponding to the «yeses» and we obtain: (S_1) 1-3-5-7-9…

[1] Ordinal series are series where the criterion is an order.
[2] We consider that S is the numerical sequence that starts at 1.

2. Construction of the series S_2. In the second step, we apply the same generative method that we used in the first. Thus, we apply the serial correspondence to the series with this other sequence: yes-no-no-yes-no-no-yes… and we obtain S_2, which would be: 1-4-7-10…
3. Construction of the series S_3. We obtain S_3 from the serial correspondence with the series: yes-no-no-no-yes-no-no-no-yes… so, S_3 is: 1-5-9… Therefore, in the nth step we obtain the succession S_n based on the serial correspondence: yes-(n-no)-yes-(n-no)-yes…

With this process a method of construction of additive numerical series has been created based on the numerical sequence.

If we combine this section with some of the previous ones we can obtain, for example, the multiplication tables in this manner:

✓ We start with the sequence of natural numbers: 1, 2, 3, 4, 5…
✓ Count two places and with the two as the first element: 2, 4, 6, 8…
✓ Count three places and with the three as the first element: 3, 6, 9, 12…
✓ Count four places and with the four as the first element: 4, 8, 12, 16…

And so on successively.

In short, all of the additive numerical series can be generated from the sequence of natural numbers using an ordinal generation method.

Discussion

Conclusions and summary

Two models have been compared: the Piagetian logical model and the skill-integration model (information processing), to investigate the development of the numerical sequence. These models address the development of the natural number in children and, in this study, we have analyzed the development of the numerical sequence from the perspective of these two models.

What is the difference between the numerical sequence and natural number?

The set of natural numbers is formed by numbers that are its elements. An important characteristic of this set is that it is in order, that its elements can be put into a sequence, one after another, and that means that every element of the set of natural numbers has two meanings: one due to the place that it occupies in the series, the ordinal aspect of the number; and the other based on the meaning that the element itself has, the cardinal aspect of the number.

The first is the basis of the use of the number to count and it is mathematically formalized by complete induction and the Peano axioms.

The second aspect provides us with the use of the number to measure a collection of discrete objects and is formalized mathematically through the equipotency of the sets.

Within the ordinal context, the most accepted construction of the system of natural numbers is Peano's Axioms, based on induction or recurrence of the «next» function and with the existence of zero as a generating element in that law of recurrence.

The Peano axioms address the ordinal aspect of the natural number, seeking sequencing of the numbers. In this construction the definition of the numerical terms is not important, but what really interests us is the «relationship of next» between them. From this point of view the construction of the numerical sequence appears from the primary concepts of logical-ordinal generators of the series, since these are associated with the successor function indicated in the second axiom.

We will define the numerical sequence as a type of series that can be generated based on logical-ordinal relationships. These definitions are given from the construction that Bertrand Russell (1903) makes of relationships of order, in turn based on the biunivocal asymmetric relationships defined by Bolzano (1851), which involve as a primary concept what he himself calls «immediately following to the side of and immediately prior to the side of».

The contribution of this paper is to analyze how development of the number in children takes place through the logical-ordinal relationships under the two models (Piagetian and information processing), which means that we have focused on the numerical sequence and overlooked the cardinal aspect of numbers. Even in the Piagetian model the logical structure of seriation has been considered without referring to the logical structure of classification or the conservation of discrete quantities; and in the model of the action of counting (information processing) we have worked on the principle of stable order with the acquisition and production of the numerical sequence, but we have not studied or analyzed the principle of cardinality (the last word of the count is the cardinal of the set), or the principle of abstraction (any collection of discrete objects is countable), or the principle of irrelevant order (the cardinal of the set is independent of the way in which the count is done). Nor have we addressed the principle of one-to-one correspondence (each and every one of the elements of the set that will be counted has one and only one corresponding numerical term) with the sole intention of calculating the cardinal of the set.

The fact that this paper describes an in-depth study of the development of the number in children through logical-ordinal relationships, ignoring every aspect of the quantity of the number, is what makes this work unprecedented and original.

The conclusions are the following:

- From the perspective of the Piagetian model we can analyze the logical structure of seriation underlying the numerical sequence.
- From the perspective of information processing, the numerical sequence is analyzed as a component of counting, but without taking into account the logical-ordinal relationships that exist between its terms. In this model, research on the functionality of counting focuses on the «quantifying operator», comparing cardinal numbers so they can subsequently be located in the sequence.
- Logical-ordinal relationships have not been the specific subject of study either in the Piagetian or the skill-integration model (information processing), which indicates that the research described in this paper is totally unprecedented and original.
- Specific tasks can be determined from the ordinal number that reflect the logical-ordinal relationships between the terms of the numerical sequence without having to treat these terms as magnitudes.

We can underline two significant consequences:

1. The basic elements that are characteristic of Piaget's logical structure of seriation are applicable to the numerical sequence and, therefore, we can take them into account in teaching on natural numbers.
2. There are exclusively ordinal tasks to assess the logical-ordinal relationships between the terms of the numerical sequence.

And with all of this we have managed to:

- Define the ordinal aspect in the education of the student on natural numbers.
- Characterize the logical relationships existing between the terms of the numerical sequence through the logical structure of seriation underlying that sequence.

Finally, it should be noted that this study poses a challenge to teachers in child education in the following regard: managing to achieve in their students the integration of the skills and routines involved in the action of counting in strategies that display some kind of logical-ordinal relationship between the numerical terms (Fernández, 2012, 2013).

Fernández and Ortiz (2008) analyze the evolution of logical-ordinal relationships in a reduced group of children selected at random, where the following guidelines are provided to teachers, taking into account for each age the skills or abilities to be achieved in function of the logical-ordinal relationships:

Class of three-year olds. Children of three years of age do not generally consider the information, so logical-ordinal skill or ability would be to *locate ordinal positions*. A specific action in the classroom, considering the given skill, would be: «Rows of objects are shown. The child has to guess the first, the fifth, etc. Reciprocally, some ordinal positions are given and the child has to ascertain to which object in the row they correspond».

Class of four-year olds. These children take the information into account, so they can develop the skill to «locate logical-ordinal positions».

Class of five-year olds. The fundamental characteristic in this class is that we no longer rely on tangible objects. Rows of objects are not shown, but instead a numerical sequence is used, since children have achieved operative success in the logical-ordinal relationships that exist between the terms in the numerical sequence, which allows these actions to be carried out in the classroom:

- Locate the next and previous of any number between 1 and 10.
- Count from a term.
- Count from one term to another.
- Count n terms from the term *a*.

In short, our research changes the basic skills in the aspect of counting. So, the skill «rote recitation of the numerical sequence» is changed for skills on the basis of the logical-ordinal relationships that occur between the numerical terms: «if such a thing happens in *a*, what happens in *b*?». Some of the skills would be:

- To determine all of the next terms from *a* up to *b* (first and last element).
- To determine each and every one of the terms in the *a*, *b* section of the sequence (between).
- To have a series-generating element on which to reason inductively (first element).
- To determine the «nexts» using the «immediate next» and reciprocally.
- To determine the «immediate next» knowing the subsequent terms.

Mastery of the numerical sequence is significant from the point of view that it concerns its systematization based on the two models studied.

The original article was received on April 23rd, 2014
The revised article was received on March 17th, 2015
The article was accepted on May 8th, 2015

References

Bermejo, V., & Bermejo, M. P. (2004). Aprendiendo a contar. In M. P. Bermejo (Coord.), *Cómo enseñar matemáticas para aprender mejor* (pp. 15-32). Madrid: CCS.

Bermejo, V., & Lago, M. O. (1991). *Aprendiendo a contar. Su relevancia en la comprensión y fundamentación de los primeros conceptos matemáticos.* Madrid: Centro de Investigación y Documentación Educativa.

Bolzano, B. (1851). *Paradoxien des Unendlichen*. Lepizig: C.H. Reclam.

Brainerd, C. J., & Gordon, L. L. (1994). Development of verbatim and gist memory for numbers. *Developmental Psychology, 30*(2), 163-77.

Brannon, E. M. (2002). The development of ordinal numerical knowledge in infancy. *Cognition, 83*, 223-240.

Cañadas, M. C., Castro, E., & Castro, E. (2012). Diferentes formas de expresar la generalización en problemas de sucesiones. *Gaceta de la Real Sociedad Matemática Española,15*(3), 561-574.

Castro, E., Cañadas, M. C., & Molina, M. (2010). El razonamiento inductivo como generador de conocimiento matemático. *Uno, 54*, 55-57.

DeSoto, M. C. (2004). Strategy choices in simple and complex addition: contributions of working memory and counting knowledge for children with mathematical disability. *Journal of Experimental Child Psychology*, *88*(2), 121-151.

Fernández, C. (2010). Análisis epistemológico de la secuencia numérica. *Revista Latinoamericana de Investigación en Matemática Educativa (RELIME)*, *13*, 59-88.

Fernández, C. (2012). Proyecto para la evaluación de la capacidad de comprender-aprender matemáticas. In F. Valdivia (Ed.), *Evaluación de los procesos de enseñanza-aprendizaje en el ámbito universitario* (pp. 157-168). Málaga: SPICUM.

Fernández, C. (2013). Contribuciones para la mejora de la educación matemática en los niños y niñas de 3 a 6 años de Andalucía. In M. C. Moreno, M. M. Gallego, & C. I. Gallego (Eds.), *Retos educativos de la cultura andaluza en una sociedad global* (pp. 543-558). Málaga: HUM-689.

Fernández, C., & Ortiz, A. (2008). La evolución del pensamiento ordinal en escolares de 3 a 6 años. *Infancia y Aprendizaje, 31*(1), 107-130. doi: 10.1174/021037008783487066

Fuson, K. C. (2000). *Pre-k to grade 2 goals and standards: Achieving 21st century mastery for all.* Paper presented at the Conference on Standards for Preschool and Kindergarten Mathematics Education, State University of New York at Buffalo, USA. Retrieved from http://www.gse.buffalo.edu/org/conference

Fuson, K., & Hall, J. (1983). The acquisition of early number word meanings: A conceptual analysis and review. In H. Ginsburg (Ed.), *The development of mathematical thinking* (pp. 49-107). New York: Academic Press.

Gálvez, G., Cosmelli, D., Cubillos, L., Leger, P., Mena Lorca, A., Tanter, E., Flores, X., Luci, X., Montoya, S., & Soto-Andrade, J. (2011). Estrategias cognitivas para el cálculo mental. *RELIME, 14*(1), 9-40.

Gelman, R., & Gallistel, C. R. (1978). *The child´s understanding of number.* Cambridge, Massachusetts: Harvard University Press.

Klhar, D., & Wallace, J. G. (1973). The role of quantification operators in the development of conservation. *Cognitive Psychology*, 4, 301-327.

Mix, K. S., Sandhofer, C. M., Moore, J. A., & Russell, C. (2012). Acquisition of the cardinal word principle: The role of input. *Early Childhood Research Quarterly, 27*(2), 274-283. doi: 10.1016/j.ecresq.2011.10.003

Muldoon, K., Lewis, C., & Towse, J. (2005). Because it's there! Why some children count, rather than infer numerical relationships. *Cognitive Development, 20*(3), 472-20.

Piaget. J. (1981). *La toma de conciencia*. Madrid: Morata.

Piaget, J., & Inhelder, B. (1976). *Génesis de las estructuras lógicas elementales: clasificaciones y seriaciones.* Buenos Aires: Guadalupe.

Piaget, J., & Szeminska, A. (1964). *La genèse du nombre chez l'enfant.* Neuchatel: Editions Delachaucx et Niestlé.

Russell, B. (1982). *Los principios de la matemática*. Madrid: Espasa Calpe. (Original work published in 1903).

Sánchez, A. B. (2013). Aprender a contar según el De Computo de Rabano Mauro. *Educación XX1*, *16*(2), 39-62. doi: 10.5944/educxx1.16.2.2632

Sánchez, M. D., & Fernandez, C. (1999). *Estudio del cardinal y el ordinal para la enseñanza y aprendizaje en educación infantil*. Paper presented at the II Congreso Mundial de Educación Infantil y Formación de Formadores, Diputación de Málaga, Spain.

Sarnecka, B. W., & Gelman, S. A (2004). Six does not just mean a lot: Preschoolers see number words as specific. *Cognition*, *92*, 329-352. doi: 10.1016/j.cognition.2003.10.001

Schaeffer, B., Eggleston, V. H., & Scott, J. L. (1974). Number development in young children. *Cognitive Psychology*, 6, 357-379.

Serrano, J. M. (2008). Acerca de la naturaleza del conocimiento matemático. *Anales de Psicología, 24*(2), 169-179.

Slusser, E. B., & Sarnecka, B. W. (2011). Find the picture of eight turtles: A link between children's counting and their knowledge of number word semantics. *Journal of Experimental Child Psychology, 110*(1), 38-51.

Song, M. J., & Ginsburg, H. P. (1988). The effect of the Korean number system on young children's counting: A natural experiment in numerical bilingualism. *International Journal of Psychology, 23*, 319-332.

Wagner, S., & Walters, J. A. (1982). A longitudinal analysis of early number concepts: From numbers to number. In G. Forman (Ed.), *Action and thought* (pp. 137-161). New York: Academic Press.