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Teaching Teachers: Roles for Mathematicians

Preparando profesores: el papel de los matemáticos

Solomon Friedberg

Department of Mathematics, Boston College, USA

Abstract

This article examines the roles that mathematicians may play in the mathematical education of teachers and why such involvement is needed. It presents some general remarks on this topic, and then describes mathematics classes directed at both future K-8 and 9-12 teachers that are offered at one specific university. Then, the role of mathematicians in the mentorship of preservice teachers is described and briefly analyzed. Lastly, professional development is considered.

Keywords: pedagogical content knowledge, preservice teacher preparation, inservice professional development, mentorship

Resumen

Este artículo examina el papel que los matemáticos pueden desempeñar en la educación matemática de los profesores y por qué esta participación es necesaria. Se presentan algunas observaciones generales sobre este tema y luego se describen las clases de matemáticas dirigidas tanto a futuros profesores de primaria como de secundaria que se imparten en una universidad específica. Luego, se describe y analiza brevemente el papel de los matemáticos en la formación de profesores. Al final del artículo, se considera el desarrollo profesional docente.

Palabras clave: conocimiento del contenido pedagógico, preparación de futuros profesores, desarrollo de profesionales en servicio, tutoría

Mathematicians and teachers: The big picture

In this article, I shall examine roles that mathematicians may play in the mathematical education of teachers and why it is advantageous to do so. Of course, this is by no means a new perspective. However, as someone who has walked the path from utter non-involvement (I never thought I'd do this when I completed my Ph.D in number theory, or that it was the role of a research-active mathematician to do so) to enthusiastic involvement, it may be of some small value to record, especially in the context of this associated conference, as mathematicians in Latin America consider ways that they may contribute to the preparation of math teachers. In this section I will present a few general remarks, then in subsequent sections I will describe mathematics classes at my university for preservice teachers directed at both the K-8 and 9-12 levels, the role of mathematicians in the mentorship of preservice teachers and what this has taught us at my institution, and lastly professional development. As we shall see, there are many promising developments, but a great deal remains to be done.

The central fact is this: teaching mathematics well requires knowledge of mathematics for teaching, that is, mathematical *pedagogical content knowledge* (PCK) (Ma, 2010). However, as Ma observed, in the context of teachers in the United States, *many (U.S.) teachers—even practicing elementary school teachers— have limited PCK of mathematics*. The situation in other countries is often similar.

For example, not all teachers can:

- Explain (in a grade-appropriate way) why the standard algorithms for addition, subtraction, multiplication and division work.
- Explain why the rule for dividing fractions

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc}$$

is true, or give examples of problems that lead to such a division.

- Explain why the graph of $y = ax+b$ is a straight line.
- Explain the geometric meaning of the "focus" of a parabola, and how this is related to the real world.
- Explain why the law of sines is true.

Addressing this shortfall requires a substantial effort, and it is here that mathematicians must play a role. Indeed, mathematicians, who work with mathematical reasoning every day and who deeply understand the interconnectedness of mathematics, are a natural group to deliver PCK. This applies at all levels, including elementary school, and to both preservice and inservice teachers. And experience demonstrates that mathematicians who are not math education specialists can contribute, from preparing future elementary school teachers to preparing future high school teachers. However, it may be helpful to remark that not every faculty member in a math department needs to do this; not everyone would be good at it or is interested. As with any other teaching, one needs specific knowledge and interest.

There is another central fact that must be kept closely in mind: Pedagogical content knowledge is not the only prerequisite to teach mathematics effectively. Being a successful math teacher requires:

- Knowledge of methods of pedagogy, for example, differentiated instruction, in the context of math.
- Classroom management skills.
- Knowledge about learning disabilities.
- Understanding of the cultures of the students in the classroom.
- Experience with students with limited proficiency in the language of instruction.
- Knowledge about child development.

So a successful program for preservice teacher development must address these teaching skills as well. In the U.S., the above topics are typically the province of colleges of Education which are to be found at many universities.[1]

I do want to note that the aspects of pedagogy listed above, taught in our colleges of Education, and PCK, the concern of mathematicians, combine in the classroom. For example,

- Differentiated instruction may be informed by PCK.
- Classroom management is impacted by PCK.

This article will focus on developing PCK. However, when mathematicians and math educators combine to teach courses, they are well-positioned to discuss these interactions. So this article gives an account of one perspective, but with the hope that it may encourage the development of a joint perspective, one that features mathematicians and math educators systematically combining their areas of expertise.

Programs for preservice teachers, K-8: Massachusetts

I would like to describe the situation in my home state of Massachusetts, as it gives a good example of what is possible.[2] In Massachusetts preservice teachers are required to take courses that develop deep knowledge of elementary math, and a partnership of mathematicians and math educators is expected in developing and delivering such courses. The requirements for state certification of an elementary teacher training program are: 3-4 college level subject matter courses (9-12 semester hours) "taught by math faculty, potentially in partnership with education faculty" (Massachusetts Department of Education, 2007).

The Massachusetts Regulations (Regulation 603 CMR 7.06[7][b], as modified on April 24, 2007) state: "Candidates shall demonstrate that they possess both fundamental computation skills and comprehensive, in-depth understanding of K-8 mathematics. They must demonstrate not only that they know how to do elementary mathematics, but that they understand and can explain to students, in multiple ways, why it makes sense" (Massachusetts Department of Education, 2007, p. 1).

The regulations come with suggested topic weightings: Numbers and Operations: 45%, Functions and Algebra: 25%, Geometry and Measurement: 20%, Statistics and Probability: 10%. Though the regulations were finalized several years before the Common Core, they are broadly consistent with it. Thus they support the current Massachusetts standards for mathematics (Massachusetts Department of Elementary and Secondary Education, 2011), which are heavily based on the Common Core State Standards for Mathematics (2011).

The Regulations moreover formulate overall goals of math classes for preservice elementary teachers:

Elementary teacher candidates are expected to attain proficiency with, as well as deep understanding of, the arithmetic, algebra, geometry, and probability that their own students will be expected to master in grades K–8. They can reach this level of knowledge if and only if they:

1. Come to view arithmetic (and algebra) as a small, unified, coherent, consistent subject that all makes sense.
2. Appreciate the importance of developing clear, explicit, grade-appropriate definitions and using logical reasoning to arrive at unambiguous conclusions.
3. Experience and do real mathematics, by struggling with problems that have multiple steps, logical challenges, and non-obvious solutions.
4. Acquire habits of mathematical thinking: Reasoning, conjecturing, visualizing, analyzing, estimating, exploring, justifying, and constantly probing with "Why?
5. Traverse many levels of abstraction: from marks on a wall to Roman numerals to place value to scientific notation; from numbers to variables to functions.
6. Gain the competence and confidence to analyze their students' mathematical thinking and engage them in productive mathematical discourse (Massachusetts Department of Education, 2007, p. 5).

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[1]   At my institution, Boston College, this is the Lynch School of Education.
[2]   By many measures, Massachusetts is the top-performing state in the United States in mathematics.

To accomplish this, the programs in Massachusetts involve both the development of subject matter knowledge and of mathematical thinking skills. I will describe each of these in turn.

First, here are some key aspects of the topics that are covered:

1. *Number and Operations.* This topic is the basis for all other school mathematics.

In teaching it in a "Math for Future Elementary Teachers" course,[3] topical emphases include the number line, place value, inverse operations, the distributive law, the standard algorithms for +, -, x, ÷, number sense, mental math and estimation. It is also important to highlight connections and examples from algebra and geometry. Indeed, arithmetic can be taught in a way that prepares students for algebra, and this is indicated throughout. This topic typically requires more than one semester.

2. *Functions and Algebra.* Algebra is the "gatekeeper subject." Success or failure in algebra plays a main role in college readiness. Future teachers must become proficient and comfortable with algebraic thinking, especially the use of variables and the solution of simple equations. Graphs, proportionality, and linearity are also important concepts. Mathematical connections are another important theme; for example algebra problems may arise from geometry (express the area of a circle as a function of its circumference), and graphing is an important connection between functions and geometry.

3. *Geometry.* Definitions are important, but so is mathematical reasoning about spacial objects. The number line, the area model for multiplication, Cartesian coordinates, the Pythagorean Theorem and its proofs, properties of triangles and other polygons, parallel lines, transformations (rotation, translation, reflection, and dilation), congruence and similarity are key topics. And the way that different geometric facts fit together is important. Students are prepared to answer questions such as: Starting with only the area formula for a rectangle, derive the area formulas for a right triangle, a non-right triangle, a parallelogram, and a trapezoid. Prove that the measures of the interior angles of a triangle sum to 180 degrees. If you triple the edges of a cube, what happens to its surface area? Its volume?

4. *Statistics and probability.* The material on this topic is mostly descriptive, as it has a secondary role prior to middle and high school. One theme is permutations, combinations, and their applications in computing probability. For example, how many different ways can the letters of "Chile" be rearranged? Will the same technique work on "Santiago?" Why or why not? What is the probability of drawing four aces in a poker hand of five cards? Other topics include descriptive statistics, measures of central tendency and dispersion, sample space, simple/compound events, independent/dependent events, and conditional probability.

Second, here are some examples of the way that mathematical thinking is developed. Mathematics is based on definitions and logical thinking (fractions, rates, etc.). Teachers in Massachusetts are asked to learn such thinking, and also that some problems are ill-posed (if half the boys and a quarter of the girls in Mrs. González's class like orange juice, what fraction of the children in Mrs. González's class like orange juice?). It follows that not every problem with numbers has an answer that can be obtained by doing something simple to the numbers.

Teachers *do* mathematics: "Doing real mathematics involves struggling with substantive problems that don't have obvious solutions; teachers need this experience, at a level they can handle. Assignments include rich activities and problems that stretch the mind, challenge the intellect, and develop mathematical thinking (in addition to exercises that provide practice and solidify skills)," (Massachusetts Department of Education, 2007). Teachers are given problems that require exploration, conjecture, and perseverance, as well as challenging multi-step word problems, counterintuitive word problems, and open problems. For example:

• A school has 961 lockers and 961 students. The first student opens every locker, the second student changes every second locker, the third student changes every third locker, and so on. Which lockers are open after every student has passed by?

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[3] Elementary teachers are expected to be well-versed in topics that follow in middle school. Similarly, one would expect middle school teachers to be well-versed in high school math as well.

- Analyze and propose a strategy for the game of NIM. Two players begin with 13 markers and take turns removing either one or two; the player who removes the last marker loses.
- Two boys started walking at sunrise, each from his village to the other's village. They met at noon. The first boy arrived in the second's village at 4:00 PM, while the second boy arrived at the first boy's village at 9:00 PM. They walked at constant rates. What time was sunrise?
- Fresh cucumbers contain 99% water by weight. One hundred kilos of cucumbers are placed in storage, but by the time they are brought to market, it is found that they contain only 98% water by weight. How much do these cucumbers now weigh?
- Estimate the number of cell phones in Santiago.
- Estimate the number of math books in Latin America.[4]

Next, I want to report on a course I taught for such preservice teachers, concerning Numbers and Operations. Here is the context at my institution[5]. The course was required of all students in the School of Education. Students were at a wide range of levels in math (all the way from "very weak" to math majors). Most did not yet have preservice teaching or other classroom experience.

In designing the course, the following features were of particular importance. First, though the course was a university math course, the school classroom was never far away. Every class meeting involved students working in groups and explaining. Many homework problems asked "if a student said X, what would you say?" Here X might be a misunderstanding, or an alternative way to do a problem that was not standard but was correct. Also, I related every topic to the math frameworks for Massachusetts and other key states. Second, PCK was discussed explicitly. Students were required to read Liping Ma's book (2010). The course required an end of semester project, designing a problem set and explaining how PCK informed the problem set. The students reported that this end of semester project was of great use to them.

Here are some things that struck me at the end of my teaching experience, or have been reported to me by my former students.

- University students are surprised by how rich the topic of Numbers and Operations is.
- Students who are weak in math experience anxiety from the beginning.
- By the end of a semester, students have a much deeper understanding of PCK and of what it means to know elementary math for teaching.
- Students appreciated that the course was not simply a lecture-format course.
- The course was also useful for future middle and high school teachers. (Many students' problems at the high school level result from their weak understanding of numbers and operations).

My course was not a "methods" course. But the methods course at Boston College, taught in the Lynch School of Education, includes math content as well, and this is coordinated with the math department.

One variable to note is the question of when future teachers should take such courses. Ultimately, at my institution we concluded that students who are already in the classroom are too concerned with classroom management and lesson plans to spend the time they need on such a body of material. So it is better for them to take this sequence before they do their preservice teaching, even though some of the topics would be more "real" if they were already in the classroom.

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[4]   These problems are adapted from Guidelines for the Mathematical Preparation of Elementary Teachers (2007).
[5]   Boston College, a highly competitive private university, receives roughly 35,000 applications each year for its freshman class of 2,250.

High school preservice math classes: Experience at Boston College

In Massachusetts, as is typical in the U.S., future high school math teachers are required to major in mathematics. A challenge for my department is that we do not have enough future high school teachers to offer a wide array of upper division courses that are directly solely at future high school teachers. And we also must secure departmental approval for such courses if they are to carry credit for the major. In Boston College:

- Future teachers, like all other math majors, must take calculus, linear algebra, introduction to abstract mathematics, abstract algebra, and real analysis, as well as a selection of upper division electives.
- There are two upper division elective courses—Mathematical Problem Solving and Euclid's Elements—that we have created with future teachers in mind. These courses are open to other math majors as well if there is room.
- Other useful courses offered by the math department that may be included in the programs of future high school teachers include Probability, Statistics, Euclidean and Non-Euclidean Geometry, and Number Theory.

The two special courses were created by different departmental faculty. Professor Ned Rosen created Mathematical Problem Solving. While the course is problem solving, the problems are chosen in order to give a review of parts of high school math (which students may not have learned thoroughly the first time). So, though it is not as systematic as an "Algebra and Trigonometry for Teachers" course would be, the course seeks to add to students' understanding of high school math. The course is also interactive, and students are required to present solutions.

Professor Mark Reader created Euclid's Elements. In this course students read Euclid (in translation). They learn the need for definitions and precision. Students must fill in the proofs of Euclid and present them in contemporary language to the class. They are required to learn mathematical typesetting using TEX. A student takes notes at each class, which are typed up by the student and provided to the class. The emphasis of the course is on mathematical reasoning, not "how far we got in the book."

Both courses have been taught by multiple faculty members over the years. The faculty share notes and teaching materials. The Education School holds them in high regard, and our graduates indicate that they were very useful.

Mentoring preservice teachers: What we have learned

An additional role for university mathematicians is in mentoring preservice teachers. Many math faculties at my institution have visited classrooms as our students do their preservice student teaching. We then meet with the student who has just taught and discuss the lesson. We see:

- Classroom management is a main issue.
- Some of our students are asked to teach very challenging classes (many students with behavioral or learning issues).
- The involvement of the supervising teacher varies widely.
- Our students do not always employ good notation, for example, avoiding use of the equal sign, not showing implications explicitly.
- The curriculum is closely prescribed by the school system. There is little room for creativity.
- Our students do not always have a good perspective on why each topic is being taught, or on the relative importance of different topics.

These observations suggest that more is needed, especially before future high school teachers go into the classroom. Ideally, we need an analogue for future high school teachers of the "Math for Elementary Teachers" sequence. This too should comprise 3-4 semesters, and should develop PCK related to high school mathematics. However, there are a number of institutional barriers to doing so. Would it carry upper division elective credit? How could we justify the resources for a small group of students? Who would teach it? Who will write the book for it? Though our special classes for future high school teachers are very useful, ideally we would do even more.

Inservice Teachers: Professional Development

In the last part of this article, I briefly discuss the role of mathematicians in the professional development of inservice teachers. Many factors suggest that mathematicians need to play a role in this arena. For example, in the U.S.:

- Most inservice elementary school teachers have not had a comprehensive 3-4 semester sequence in Math for Elementary Teachers. They received their license as a teacher when such a sequence was not required. Indeed, a good number have had very little training in math.
- Some middle school teachers had elementary licenses or prepared to be elementary teachers and have now moved up to middle school.
- In well-off districts, high school teachers may well have been math majors. But what they learned, how well they learned it, and how it pertains to teaching high school math are all not clear (Wu, 2011).
- In poorer districts, non-trivial numbers of high school teachers were not math majors. They may be certified as "highly qualified" by a school administrator who does not know math, and may still lack important subject matter knowledge.
- Teacher certification and licensure requirements have limited effectiveness in keeping teachers who do not know their mathematics out of the math classroom (Tests may be easy and allow teachers to get many things wrong and still pass).
- Training in statistics may well be lacking.

The uneven preparation of inservice math teachers suggests that it would be helpful to have books and other teaching materials (including online materials) that are "robust," that is, that can be used effectively by math teachers even if they are lacking in content knowledge. The Singapore elementary school materials are a good example, and were designed with this possibility in mind. In addition, we need systematic Professional Development centered on math content. Mathematicians have a valuable role to play in creating such development opportunities. Some desired aspects of inservice professional development are:

- Opportunities for elementary and middle school teachers to systematically develop PCK of elementary and middle school math.
- More content rich courses for high school teachers focusing on high school content knowledge.
- Opportunities for all teachers to have "mathematical experiences."
- Subject matter knowledge of statistics that is consistent with the latest standards and expectations.

There are many promising recent changes in professional development in the U.S. These include the significant involvement of mathematicians in PD. Of particular note are efforts to develop systematic courses for elementary teachers (Vermont Mathematic Initiative, 2012), efforts to provide content rich courses for middle and high school teachers (for example, the programs at the University of Chicago and Math For America, the Noyce Master Teacher track), and efforts to give teachers mathematical experiences (for example, PROMYS for Teachers, Focus on Math II). It seems likely that the U.S. is in the process of dramatically improving its professional development opportunities for math teachers. And in doing so, we are creating a community of mathematicians interested in education.

A final aspect that is promising is the development of an international community of mathematicians who are interested in mathematics education, and of mathematics educators who are interested in partnering with mathematicians. This conference in Latin America is a wonderful example.

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